Exercise 11

Solve the differential equation.

$$2\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - y = 0$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad \frac{dy}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2y}{dt^2} = r^2e^{rt}$$

Plug these formulas into the ODE.

$$2(r^2e^{rt}) + 2(re^{rt}) - e^{rt} = 0$$

Divide both sides by e^{rt}.

$$2r^2 + 2r - 1 = 0$$

Solve for r.

$$r = \frac{-2 \pm \sqrt{4 - 4(2)(-1)}}{2(2)} = \frac{-2 \pm \sqrt{12}}{4} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

$$r = \left\{-\frac{1}{2} - \frac{\sqrt{3}}{2}, -\frac{1}{2} + \frac{\sqrt{3}}{2}\right\}$$

Two solutions to the ODE are $e^{(-1/2-\sqrt{3}/2)t}$ and $e^{(-1/2+\sqrt{3}/2)t}$. By the principle of superposition, then,

$$y(t) = C_1e^{(-1/2-\sqrt{3}/2)t} + C_2e^{(-1/2+\sqrt{3}/2)t},$$

where C_1 and C_2 are arbitrary constants.